Chapter 7 Statistical Intervals

§7.1 Introduction to Confidence Intervals

Example: X is a random variable with $\mu = E[X]$
$\sigma^2 = Var[X]$

Sample X multiple times to get IID observations

$$X_1, X_2, \dots, X_n$$

Combine to get point estimator $\bar{X}$ for μ

Know: $E[\bar{X}] = \mu$ ("unbiased estimator")

$Var[\bar{X}] = \sigma^2/n$

↪ "standard error" $\hat{\sigma}_{\bar{X}} = \sigma/\sqrt{n}$

Also, if $n > 30$ then

$$\bar{X} \approx \text{Normal}(\mu, \sigma/\sqrt{n})$$

(by "Central Limit Thm" §5.4)

Plugging in observations gives point estimate

$\bar{x} \approx \mu$ (because $E[\bar{X}] = \mu$)

but $\bar{x} \neq \mu$ ← Idea: "Fatten" $\bar{x}$ into an interval $(\bar{x} - \varepsilon, \bar{x} + \varepsilon)$

Def: A confidence interval (CI) is

$$(\bar{x} - \varepsilon, \bar{x} + \varepsilon) = I$$

where $P(\mu \in I)$ is "high".

$I =$ 

μ is in here... somewhere...

Notation: $I_\alpha = (\bar{x} - \varepsilon_\alpha, \bar{x} + \varepsilon_\alpha)$ so that

$$P(\mu \in I_\alpha) = 1 - \alpha$$

$$P(\bar{x} - \varepsilon_\alpha < \mu < \bar{x} + \varepsilon_\alpha) = P(|\mu - \bar{x}| < \varepsilon_\alpha)$$

Often we write just

"$\mu = \bar{x} \pm \varepsilon_\alpha$ (with prob. $1-\alpha$)"

For example the 95% confidence interval is

$$I_{.05} = (\bar{x} - \varepsilon_{.05}, \bar{x} + \varepsilon_{.05})$$

To compute $\varepsilon_{.05}$ we find a critical value.

$$P(|\mu - \bar{X}| < \varepsilon_{.05}) = .95$$

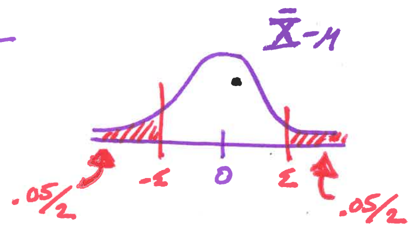
$$\hookrightarrow P(|\bar{X} - \mu| > \varepsilon_{.05}) = 1 - .95 = .05$$

Recall: $\bar{X} \approx \text{Normal}(\mu, \sigma/\sqrt{n})$

$\bar{X} - \mu \approx \text{Normal}(0, \sigma/\sqrt{n})$

$P(|\bar{X}-\mu| > \varepsilon_{.05}) = .05$

$$\begin{cases} P(\bar{X}-\mu < -\varepsilon) = .05/2 \\ P(\bar{X}-\mu > \varepsilon) = .05/2 \end{cases}$$



so $\varepsilon_{.05} = -\text{qnorm}(.05/2, 0, \sigma/\sqrt{n})$

Often we prefer to work using <u>standard normal</u>

$\bar{X}-\mu \approx \text{Normal}(0, \sigma/\sqrt{n})$ $\quad$ $\text{qnorm}(\alpha/2, 0, \sigma/\sqrt{n})$

$\updownarrow$ $\quad\quad$ $\|$

$\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \approx \text{Normal}(0, 1)$ $\quad$ $\sigma/\sqrt{n} \cdot \text{qnorm}(\alpha/2, 0, 1)$

$(1-\alpha)$ <u>Confidence Interval</u> for μ is

$$\mu = \bar{x} \pm \text{qnorm}(\alpha/2, 0, \sigma/\sqrt{n})$$

$$= \bar{x} \pm \underbrace{\sigma/\sqrt{n} \cdot \text{qnorm}(\alpha/2, 0, 1)}_{\text{This is often written "}z_{\alpha/2}\text{"}}$$

(z_α is <u>critical value</u> $P(Z > z_\alpha) = \alpha$)

<u>Example</u>: Make 100 measurements of random variable X with $\sigma = 20$. If sample mean is $\bar{x} = 30$ then what is 95% confidence interval for μ? 99%?

Know $\bar{X} \approx \text{Normal}(\mu, \underbrace{20/\sqrt{100}}_{= 20/10 = 2})$

95% CI is $\bar{x} \pm \text{qnorm}(.05/2, 0, 2)$

$\boxed{30 \pm 3.92}$ ← Note: I removed a - sign...

99% CI is $\bar{x} \pm \text{qnorm}(.01/2, 0, 2)$

$\boxed{30 \pm 5.15}$

People who do confidence interval calculations a <u>lot</u> will memorize a few values of z_α

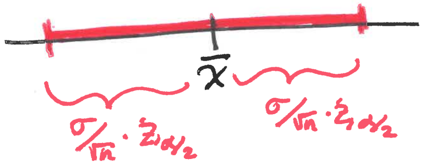
$z_{.025} = -\text{qnorm}(.025, 0, 1) \approx 1.96$ ← for 95% CI

$z_{.005} = -\text{qnorm}(.005, 0, 1) \approx 2.58$ ← for 99% CI

<u>Example</u>: 95% CI is $\mu = \bar{x} \pm (1.96)\,\sigma/\sqrt{n}$

Note: $\bar{x} \pm \sigma/\sqrt{n}\, z_{\alpha/2}$ is the interval

$$\left(\bar{x} - \sigma/\sqrt{n} \cdot z_{\alpha/2},\ \bar{x} + \sigma/\sqrt{n} \cdot z_{\alpha/2}\right)$$



Width of interval is

$$w = 2\left(\sigma/\sqrt{n} \cdot z_{\alpha/2}\right)$$

- Bigger $n \implies$ Smaller interval
  (More samples)
- Bigger $(1-\alpha) \implies$ Larger interval
  (Higher confidence)

If you have a target width, you can choose n (number of sample measurements) to make the CI appropriately narrow.

$$w = 2\left(\sigma/\sqrt{n} \cdot z_{\alpha/2}\right) \implies n = \left(2 \frac{\sigma}{w} z_{\alpha/2}\right)^2$$

Example: If X has $\sigma = 20$ how many samples are needed to have 95% CI with width 3?

$$n = \left(2\, \frac{20}{3} (1.96)\right)^2$$

$$\approx 682.95$$

$$\boxed{n = 683}$$ (because n is an integer)

... That is a *lot* of samples...

Big Question: Why not just use $z_{.025} \approx 2$?

1.96 isn't far from 2...

And "68–95–99.7" Rule says

"Approximately 95% of all probability is within 2 std. dev. of mean."

$\Updownarrow$

"$P(|X-\mu| < 2\sigma) \approx 95\%$"

(i.e. $z_{.025} \approx 2$)

(Using 2 instead of 1.96 would compute 95.45% CI, which is hardly different...)